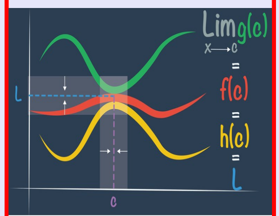
Calculus I

Lecture 19

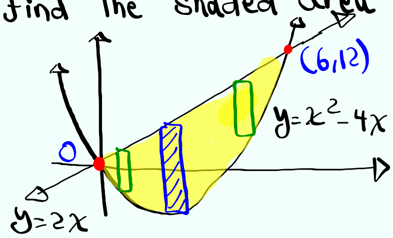


Feb 19-8:47 AM

Open notes

Class QZ 17

Find the shaded area below



Top $2x$, Bottom x^2-4x

$x^2-4x=2x$

$x^2-6x=0$

$x(x-6)=0$

$x=0 \quad x=6$

$A=\int_0^6 \left[2x-(x^2-4x)\right]dx$

$2x-x^2+4x$

$=\int_0^6 (6x-x^2)dx$

$=\left(3x^2-\frac{x^3}{3}\right)\Big|_0^6$

$=3(6^2)-\frac{6^3}{3}=$

$=3\cdot 36-\frac{216}{3}$

$=108-72$

$=\boxed{36}$ ✓

Feb 4-12:01 PM

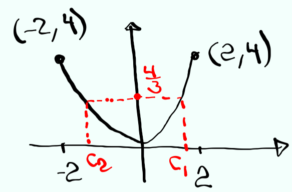
If $f(x)$ is a cont. function over $[a,b]$, then the average value of $f(x)$ is given by $\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$

ex: Find f_{ave} for $\boxed{f(x) = x^2}$ on $[-2, 2]$.
a, b

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$ provided that $f(x)$ is cont. on $[a,b]$.

$f_{ave} = \frac{1}{2-(-2)}\int_{-2}^{2} x^2\,dx$ ← even function

$= \frac{1}{4} \cdot 2\int_0^2 x^2\,dx = \frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^2 = \frac{8}{6} = \boxed{\frac{4}{3}}$



Feb 5-8:13 AM

Find f_{ave} for $\boxed{f(x) = x^3}$ over $[-3,3]$.
cont. $(-\infty,\infty)$

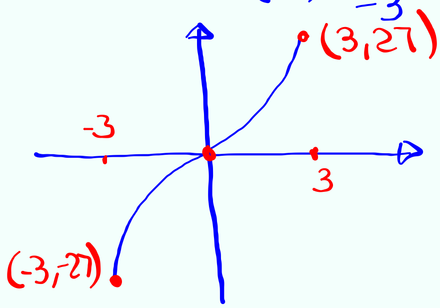
$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$ provided that $f(x)$ is cont. on $[a,b]$.

$f_{ave} = \frac{1}{3-(-3)}\int_{-3}^{3} x^3\,dx = \frac{1}{6}\int_{-3}^{3} x^3\,dx = \frac{1}{6}\cdot 0 = \boxed{0}$

odd function

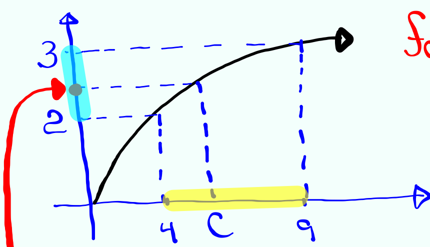
$= \frac{1}{6}\,\frac{x^4}{4}\Big|_{-3}^{3} = \frac{1}{24}\left(3^4 - (-3)^4\right)$

$= \frac{1}{24}(81 - 81) =$



Feb 5-8:21 AM

Find f_{ave} for $f(x)=\sqrt{x}$ over $[4,9]$.



$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$=\frac{1}{9-4}\int_4^9 \sqrt{x}\,dx \qquad x^{1/2}$$

$$=\frac{1}{5}\cdot\frac{x^{3/2}}{3/2}\Big|_4^9$$

$$=\frac{2}{15}\,x\sqrt{x}\Big|_4^9$$

$$=\frac{2}{15}\left[9\sqrt{9}-4\sqrt{4}\right]$$

$$=\frac{2}{15}\left[27-8\right]$$

$$=\boxed{\frac{38}{15}}\approx 2.5$$

$f(c)=f_{ave}$

$\sqrt{c}=2.5$

$c=(2.5)^2$

$\boxed{c\approx 6.25}$

Feb 5-8:26 AM

Find f_{ave} for $\underbrace{f(x)=x^2(x^3+1)^4}_{\text{Cont. }(-\infty,\infty)}$ over $[0,2]$.

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\,dx$$

$u=x^3+1$

$du=3x^2dx$

$\frac{du}{3}=x^2dx$

$x=0 \quad u=1$

$x=2 \quad u=9$

$$=\frac{1}{2-0}\int_0^2 x^2(x^3+1)^4\,dx$$

$$=\frac{1}{2}\int_1^9 u^4\,\frac{du}{3}$$

$$=\frac{1}{6}\,\frac{u^5}{5}\Big|_1^9=\frac{1}{30}\left(9^5-1^5\right)$$

$$=\frac{1}{30}\cdot 59048=\boxed{\frac{29524}{15}}$$

Feb 5-8:33 AM

Find f_{ave} for $f(x) = \cos^4 x \sin x$ over $[0, \pi]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$x = 0 \quad u = 1$

$x = \pi \quad u = -1$

$$= \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 x \sin x\,dx$$

$$= \frac{1}{\pi}\int_1^{-1} u^4 \cdot -du$$

$$= \frac{1}{\pi} \cdot -\int_{-1}^{1} u^4 \cdot -du = \frac{1}{\pi}\int_{-1}^{1} u^4\,du$$

$$= \frac{1}{\pi} \cdot 2\int_0^1 u^4\,du = \frac{2}{\pi} \cdot \frac{u^5}{5}\Big|_0^1$$

$$= \boxed{\frac{2}{5\pi}}$$

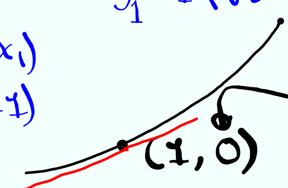
Feb 5-8:41 AM

Find equation of the tan. line to the graph of $f(x) = \int_1^{x^6} \frac{1}{1+\sqrt[3]{t}}\,dt$ at $x = 1$.

$y - y_1 = m(x - x_1)$

$y - 0 = 3(x - 1)$

$\boxed{y = 3x - 3}$



$f(1) = \int_1^1 \cdots\cdots = 0$

$$m = f'(1) = \frac{6(1)^5}{1 + 1^2} = \frac{6}{2} = 3$$

$$f(x) = \int_1^{x^6} \frac{1}{1+\sqrt[3]{t}}\,dx$$

$$f'(x) = \frac{1}{1 + \sqrt[3]{x^6}} \cdot 6x^5 - \frac{1}{1 + \sqrt[3]{1}} \cdot 0$$

$$f'(x) = \frac{6x^5}{1 + x^2}$$

Feb 5-8:49 AM

Find $f'(x)$ for $f(x) = \int_{\boxed{2x}}^{\boxed{x^2}} \frac{t}{1+t^5}\, dt.$

$$f'(x) = \frac{x^2}{1+(x^2)^5} \cdot 2x - \frac{2x}{1+(2x)^5} \cdot 2$$

$$= \frac{2x^3}{1+x^{10}} - \frac{4x}{1+32x^5}$$

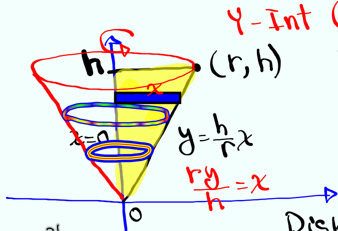
Feb 5-8:54 AM

Draw the enclosed region by $\underline{x=0}$, $\underline{y=h}$, and $\underline{y=\frac{h}{r}x}$.

Y-axis.

horizontal line

line with slope $\frac{h}{r}$ and Y-Int $(0,0)$.



Rotate this region by Y-axis.

Find the volume.

Disk Method

$V = \frac{\pi r^2 h}{3}$

$$V = \int_0^h \pi\left[\frac{ry}{h}\right]^2 dy = \frac{\pi \cdot r^2}{h^2}\int_0^h y^2\, dy$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{y^3}{3}\Big|_0^h = \frac{\pi r^2}{3h^2}\, y^3\Big|_0^h$$

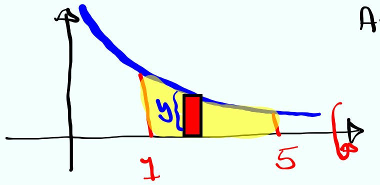
$$= \frac{\pi r^2}{3h^2} \cdot (h^3 - 0) = \frac{\pi r^2 \cancel{h^3}\, h}{3\cancel{h^2}}$$

$$= \frac{\pi r^2 h}{3}$$

Feb 5-8:59 AM

Draw the region bounded by

$y = \frac{1}{x}$, $y=0$, $x=1$, and $x=5$.



$$\text{Area} = \int_1^5 \left(\frac{1}{x} - 0\right) dx$$

$$= \int_1^5 \frac{1}{x}\,dx = \ln x \Big|_1^5$$

$$= \ln 5 - \cancel{\ln 1}^{\,0}$$

$$= \boxed{\ln 5}$$

Rotate by x-axis.

Disk

$$V = \int_1^5 \pi \left[\frac{1}{x}\right]^2 dx = \pi \int_1^5 x^{-2}\,dx$$

$$= \pi \frac{x^{-1}}{-1}\Big|_1^5 = -\pi \cdot \frac{1}{x}\Big|_1^5$$

$$= -\pi\left(\frac{1}{5} - 1\right)$$

$$= -\pi \cdot -\frac{4}{5} = \boxed{\frac{4\pi}{5}}$$

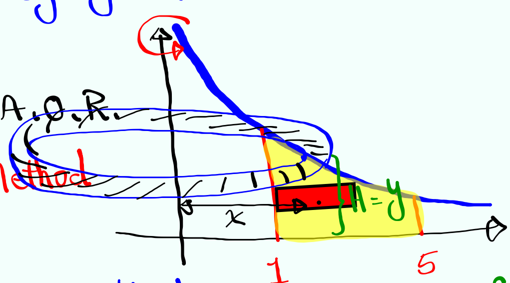
Feb 5-9:11 AM

Now rotate by y-axis

Ref. Rect. is

Parallel to A.O.R.

Use Shell Method



$$V = \int_a^b 2\pi \cdot D \cdot H\,dx$$

Height of Ref. Rect.

Distance of Ref. Rect. from A.O.R.

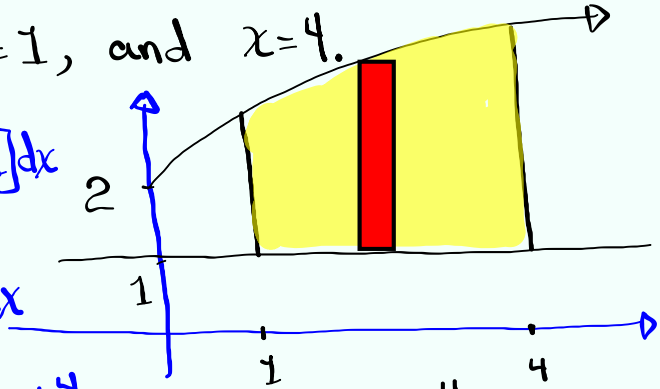
$$= \int_1^5 2\pi \cdot \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx = 2\pi \int_1^5 dx$$

$$= 2\pi \cdot x \Big|_1^5 = 2\pi \cdot 4$$

$$= \boxed{8\pi}$$

Feb 5-9:20 AM

Draw the region bounded by $y = 2 + \sqrt{x}$, $y = 1$, $x = 1$, and $x = 4$.



$$\text{Area} = \int_1^4 [2 + \sqrt{x} - 1]\,dx$$

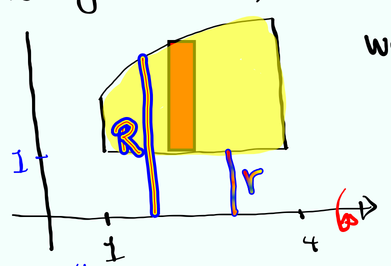
$$= \int_1^4 [1 + x^{1/2}]\,dx$$

$$= \left(x + \frac{x^{3/2}}{3/2}\right)\Bigg|_1^4 = \left(x + \frac{2}{3}x\sqrt{x}\right)\Bigg|_1^4 = \left(4 + \frac{2}{3}\cdot 4\sqrt{4}\right) - \left(1 + \frac{2}{3}\right)$$

$$= 4 + \frac{16}{3} - 1 - \frac{2}{3} = 3 + \frac{14}{3} = \boxed{\frac{23}{3}}$$

Feb 5-9:49 AM

Rotate by x-axis, find Volume



Washer Method

$R = 2 + \sqrt{x}$

$r = 1$

$$V = \int_1^4 \pi\left[R^2 - r^2\right]dx = \pi \int_1^4 \left((2+\sqrt{x})^2 - 1^2\right)dx$$

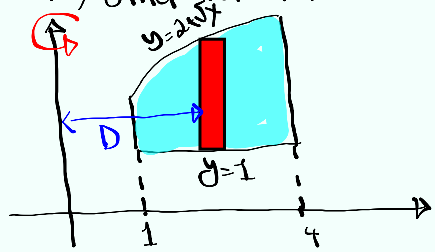
$$= \pi \int_1^4 \left[4 + 4\sqrt{x} + x - 1\right]dx$$

$$= \pi \int_1^4 \left[3 + 4x^{1/2} + x\right]dx$$

$$= \pi\left(3x + \frac{4x^{3/2}}{3/2} + \frac{x^2}{2}\right)\Bigg|_1^4 = \boxed{}$$

Feb 5-9:56 AM

Rotate about y-axis, find Volume.



Shell Method

$$V=\int_1^4 2\pi\cdot D\cdot H\,dx$$

Distance of Ref. Rect. from A.O.R.

$D=x$

Height of Ref. Rect. $H=$ Top $-$ Bottom

$=2\sqrt{x}-1$

$=1+\sqrt{x}$

$$V=\int_1^4 2\pi x(1+\sqrt{x})\,dx$$

$$=2\pi\int_1^4\left[x+x^{3/2}\right]dx=2\pi\left[\frac{x^2}{2}+\frac{x^{5/2}}{5/2}\right]\Bigg|_1^4$$

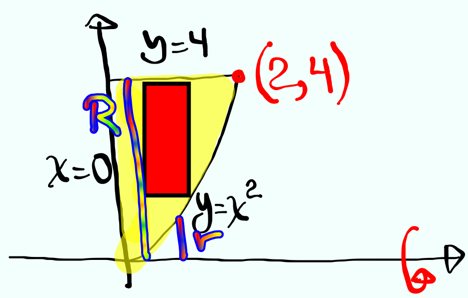
$$=\boxed{}$$

Feb 5-10:02 AM

Draw the region bounded by

$y=x^2,\ y=4,\ x=0$ in QI.

$$A=\int_0^2(4-x^2)\,dx$$



Rotate by x-axis.

Find Volume.

washer $R=4$, $r=x^2$

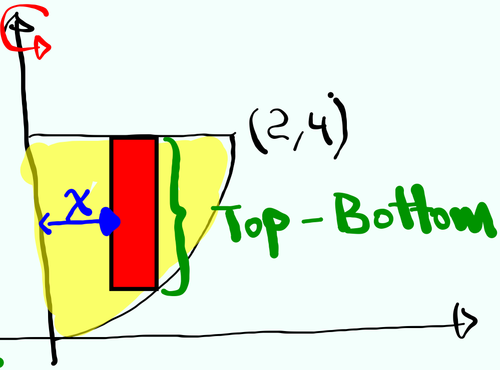
$$V=\int_0^2\pi\left[4^2-(x^2)^2\right]dx=\pi\int_0^2(16-x^4)\,dx=\boxed{}$$

Feb 5-10:08 AM

Rotate about y-axis. Find Volume

Shell

$$V=\int_0^2 2\pi\, D\, H\, dx$$



$$=\int_0^2 2\pi\, x\,(4-x^2)\,dx$$

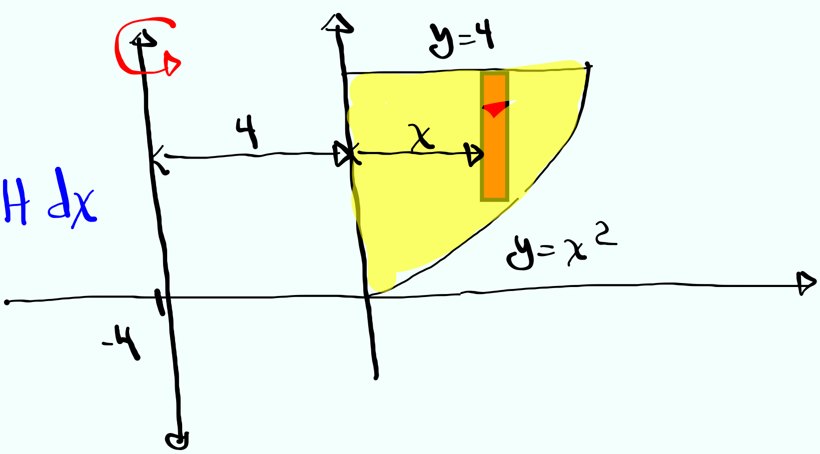
$$=2\pi\int_0^2 (4x-x^3)\,dx=2\pi\left[\frac{4x^2}{2}-\frac{x^4}{4}\right]\Bigg|_0^2=\boxed{}$$

Feb 5-10:15 AM

Rotate about $x=-4$, Find Volume.

Shell

$$V=\int_1^4 2\pi\, D\, H\, dx$$

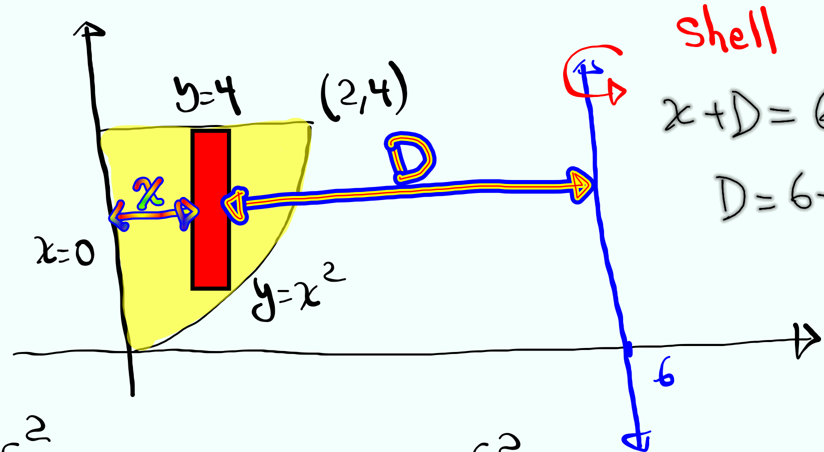


$$V=\int_0^2 2\pi(x+4)(4-x^2)\,dx$$

$$=2\pi\int_0^2 (4x-x^3+16-4x^2)\,dx=\boxed{}$$

Feb 5-10:20 AM

Rotate about $x=6$, find the volume



Shell

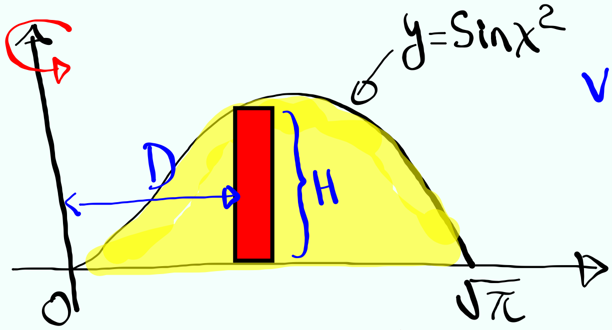
$x+D=6$

$D=6-x$

$$V=\int_0^2 2\pi\, D\, H\, dx = 2\pi\int_0^2 (6-x)\cdot(4-x^2)\,dx$$

Feb 5-10:26 AM

Rotate the region given below by y-axis, find the volume



Shell

$$V=\int_0^{\sqrt{\pi}} 2\pi\, x \sin \overset{u}{x^2}\, \overset{\frac{du}{2}}{dx}$$

$u=x^2$

$du=2x\,dx$

$x=0 \quad u=0$

$x=\sqrt{\pi} \quad u=\pi$

$$=\int_0^{\pi} \pi \sin u\, du$$

$$=\pi\cdot -\cos u\Big|_0^{\pi} = \boxed{}$$

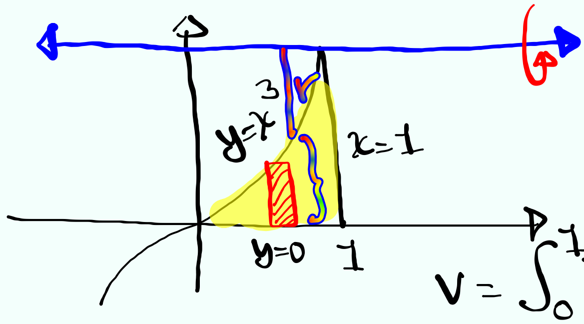
Feb 5-10:32 AM

Rotate by $y=1$ the enclosed region with $y=x^3$, $y=0$, and $x=1$.

$r+y=1$

$r=1-y$

$=1-x^3$

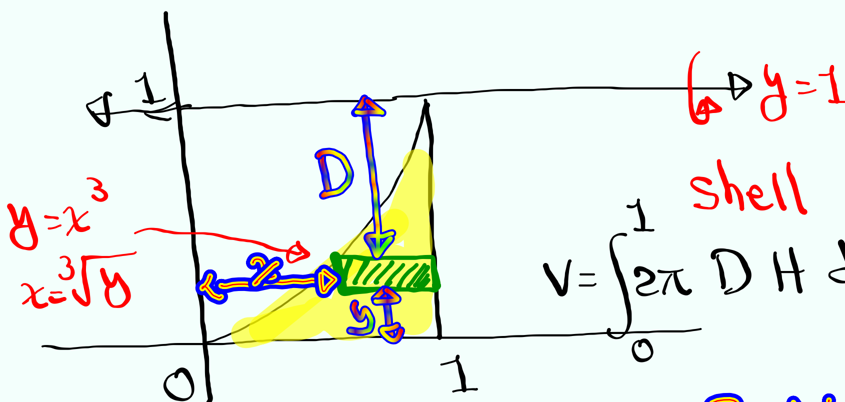


washer

$R=1$

$$V=\int_0^1 \pi\left[1^2-(1-x^3)^2\right]dx$$

Feb 5-10:38 AM



shell

$$V=\int_0^1 2\pi\, D\, H\, dy$$

$D+y=1$ $\quad D=1-y$

$H=$ Right $-$ left

$=1-\sqrt[3]{y}$

$$V=\int_0^1 2\pi(1-y)(1-\sqrt[3]{y})\,dy$$

Feb 5-10:46 AM

Draw the region bounded by

$x = 1+(y-2)^2$ and $x = 2$.

Parabola
opens Right
Vertex $(1,2)$

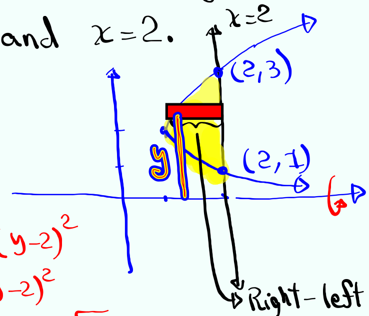
if $x=2$ $\quad 2 = 1+(y-2)^2$

$1=(y-2)^2$

$y-2=\pm\sqrt{1}$

$y=2\pm 1 \quad y=3$

$y=1$



Right − left

$2 - x_{curve}$

$2-1-(y-2)^2$

$$\text{Area} = \int_1^3 \left(1-(y-2)^2\right)dy = \boxed{}$$

Rotate by x-axis, find Volume.

Shell

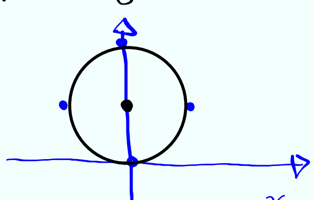
$$V=\int_1^3 2\pi \underbrace{D}_{y}\, \overbrace{H}^{2-1-(y-2)^2}\, dy = 2\pi\int_1^3 y\left(1-(y-2)^2\right)dy = \boxed{}$$

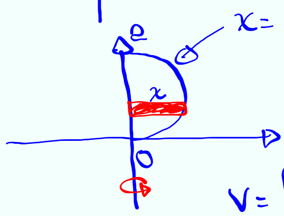
Feb 5-10:51 AM

Circle $R=1$, Center $(0,1)$

Rotate the region bounded by $x^2+(y-1)^2=1$ by the y-axis. Find Volume.

$(x-h)^2+(y-k)^2=R^2$





$x^2+(y-1)^2=1$

$x^2=1-(y-1)^2$

$x=\sqrt{1-(y-1)^2}$

Disk

$$V=\int_0^2 \pi\left[\sqrt{1-(y-1)^2}\right]^2 dy$$

$$=\pi\int_0^2 \left(1-(y-1)^2\right)dy = \boxed{}$$

Feb 5-11:04 AM